

On the geometry of a class of invariant measures and a problem of Aldous

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Abstract

In his survey [4] of notions of exchangeability, Aldous introduced a form of exchangeability corresponding to the symmetries of the infinite discrete cube, and asked whether these exchangeable probability measures enjoy a representation theorem similar to those for exchangeable sequences [11], arrays [12, 13, 1, 2] and set-indexed families [15]. In this note we prove that, whereas the known representation theorems for different classes of partially exchangeable probability measure imply that the compact convex set of such measures is a Bauer simplex (that is, its subset of extreme points is closed), in the case of cube-exchangeability it is a copy of the Poulsen simplex (in which the extreme points are dense). This follows from the arguments used by Glasner and Weiss' for their characterization in [9] of property (T) in terms of the geometry of the simplex of invariant measures for associated generalized Bernoulli actions.

The emergence of this Poulsen simplex suggests that, if a representation theorem for these processes is available at all, it must take a very different form from the case of set-indexed exchangeable families.

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1 Introduction

Suppose that K is a standard Borel space with σ -algebra Σ_K , that T is a countably infinite set and Γ a group of permutations of T and that μ is a probability measure on the (standard Borel) product measurable space $(K^T, \Sigma_K^{\otimes T})$. Let us also always assume that Γ has only infinite orbits in T . Then following Aldous [4] we shall write that μ is (T, Γ) -**exchangeable** if it is invariant under the (contravariant) coordinate-permuting action τ of Γ on K^T given by

$$\tau^\gamma((\omega_t)_{t \in T}) := (\omega_{\gamma(t)})_{t \in T},$$

which is clearly measurable and invertible. We write $\text{Pr}^\Gamma K^T$ for the set of all such exchangeable probability measures. We shall sometimes refer to the index-set action $\Gamma \curvearrowright T$ as an **exchangeability context**.

The prototypical examples of exchangeability are arguably those of **hypergraph exchangeability**, for which $T = \binom{S}{k}$, the set of all k -subsets of a countably infinite ‘vertex set’ S , and $\Gamma = \text{Sym}_0(S)$, the group of all finitely-supported permutations of S acting on T by vertex-permutations. In this case we can interpret μ as the law of a random ‘colouring’ of the complete k -uniform hypergraph on S by points from the space K of ‘colours’.

In the simplest case $k = 1$ (so $T = S$), the precise structure of all possible hypergraph-exchangeable measures follows from classical theorems of de Finetti and Hewitt & Savage (see, for example, [11]). More recently, the case of more general k was studied by Hoover [12, 13], Aldous [1, 2, 4] and Kallenberg [15], along with a number of further extensions that are still closely related to this hypergraph-colouring setting, leading to a more elaborate conception of ‘exchangeability theory’. It turns out that in these contexts too the exchangeable probability measures admit a more-or-less complete structural description, albeit involving increasingly complicated ingredients as k increases: they can all be represented as images of certain other exchangeable processes whose laws take a particular simple form. We refer the reader to [6] for a recent survey of these results and their relations to various questions in graph and hypergraph theory, and to the survey [4] of Aldous for a general introduction to a broader range of exchangeability contexts and to

the recent book of Kallenberg [17] for the modern state of the theory.

We will not recount the details of these representation theorems here. Rather, our interest lies in a different exchangeability context, proposed by Aldous as a possible object of further study in Section 16 of [4]: that of **cube-exchangeability**. Let $\mathbb{F}_2 = \{0, 1\}$ be the field of two elements, and in the d -dimensional vector space \mathbb{F}_2^d over \mathbb{F}_2 write e_1, e_2, \dots, e_d for the standard basis. Now take T to be the set $\mathbb{F}_2^{\oplus \mathbb{N}}$ of all strings of 0s and 1s with only finitely many of the latter, and let Γ be the group of permutations of T generated by finitely-supported permutations of the underlying copy of \mathbb{N} together with all ‘bit-flips’: $\sigma_i : \mathbb{F}_2^{\oplus \mathbb{N}} \rightarrow \mathbb{F}_2^{\oplus \mathbb{N}} : x \mapsto x + e_i$.

In this context, given any standard Borel space K we shall call a probability measure μ on K^T **cube-exchangeable** if it is invariant under the coordinate-permuting action of the above group Γ . Note that we may describe this group as follows: T may be written as the increasing union $\bigcup_{n \geq 1} T_n$ of the discrete cubes $T_n := \mathbb{F}_2^n$, and now (bearing in mind our restriction to *finitely-supported* permutations of \mathbb{N}) every member $g \in \Gamma$ actually maps T_n onto itself for all sufficiently large n . It is easy to see that in this case a permutation of T_n is induced by a member of Γ if and only if it is an isometry of T_n when this latter is identified with the n -dimensional Hamming cube $\{0, 1\}^n$. For this reason we shall refer to Γ as the group of **isometries of the infinite-dimensional discrete cube** and denote it by $\text{Isom } \mathbb{F}_2^{\oplus \mathbb{N}}$. Note that, as in the setting of hypergraph-exchangeability, the acting group Γ is locally finite (that is, any finite collection of its elements generates a finite subgroup); but *unlike* in that setting most elements of the group (to be precise, all that involve a nontrivial translation) do move infinitely many points of T .

In view of the success of the basic theory of hypergraph-exchangeability, Aldous asked in [4] whether a similarly precise structural description is available for the class of cube-exchangeable probability measures. In this note we will provide some evidence to suggest that such a structural description may *not* be available in this context — at least not in the very explicit form familiar from the hypergraph setting — in the following ‘soft’ sense. First, we note that, provided Γ is amenable (as it certainly is in our examples), the basic representation theorems for hypergraph exchangeable laws fall into a certain quite general pattern, and that this pattern has, in particular, the consequence that for a compact metric K the set of all extreme points (that is, ergodic members) of $\text{Pr}^\Gamma K^T$ forms a closed subgroup of this compact convex set in the vague topology; that is, this convex set is a **Bauer simplex**. On the other hand, we will show that provided K is not a singleton, this set $\text{Pr}^\Gamma K^T$ in the case of cube-exchangeability has the very different property of being a copy of the **Poulsen simplex**: its extreme points form a vaguely dense subset. This suggests that any representation theorem describing this set, if one is available, must take a rather different form from the earlier set-indexed examples.

Remark on notation

Our basic combinatorial and measure-theoretic notation is completely standard. If (X, ρ) is a metric space, $x, y \in X$ and $\varepsilon > 0$, we shall sometimes write $x \approx_\varepsilon y$ in place of $\rho(x, y) < \varepsilon$ when the particular metric ρ is understood.

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2 The form of previous representation theorems for exchangeable measures

In this section we introduce a general template for a kind of representation theorem for exchangeable laws, which in particular characterizes the basic representation theorems for the cluster of variations on hypergraph-exchangeability.

These theorems all focus on representing an arbitrary (T, Γ) -exchangeable process as an image (in a suitable sense) of another exchangeable process (possibly with a different index set) for which the different random variables are all mutually independent.

Definition 2.1 (Ingredients). *Let $\Gamma \curvearrowright T$ be an exchangeability context and K a fixed compact metric space. By a list of **representation data** we understand:*

- *a sequence of auxiliary index sets T_1, T_2, \dots each endowed with some action $\Gamma \curvearrowright T_i$ that has only infinite orbits;*
- *a disjoint sequence of **dependency maps** $\phi_i : T \rightarrow \binom{T_i}{<\infty}$ that are Γ -covariant, in that $\phi_i(\gamma(t)) = \gamma(\phi_i(t))$;*
- *and a family of probability kernels*

$$\kappa_t : [0, 1] \times [0, 1]^{\phi_1(t)} \times [0, 1]^{\phi_2(t)} \times \dots \rightsquigarrow K$$

that is Γ -covariant, in that $\kappa_{\gamma(t)} = \kappa_t \circ (\text{id}_{[0,1]} \times \tau_1^\gamma \times \dots)$.

Given ingredients as above, we denote by $\kappa^{(T)}$ the kernel $[0, 1] \times [0, 1]^{T_1} \times [0, 1]^{T_2} \times \dots \rightsquigarrow K^T$ given by

$$\kappa^{(T)}(x_0, \mathbf{x}_1, \dots, \cdot) = \bigotimes_{t \in T} \kappa_t(x_0, \mathbf{x}_1|_{\phi_1(t)}, \dots, \cdot).$$

Of the conditions on the data introduced above, perhaps the least intuitive is that the actions $\Gamma \curvearrowright T_i$ may not have finite orbits (although it certainly holds in the case of hypergraph exchangeability); we shall later need to play this off against the finiteness of the sets $\phi_i(t)$, and it does hold for the case of hypergraph-exchangeability.

Now and henceforth we will denote by μ_L Lebesgue measure on the unit interval $[0, 1]$, and by the shorthand $\mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots}$ the product measure $\mu_L \otimes \mu_L^{\otimes T_1} \otimes \mu_L^{\otimes T_2} \otimes \dots$.

Definition 2.2 (Representability). *Given an exchangeability context $\Gamma \curvearrowright T$ and a compact metric space K , we shall say that a (T, Γ) -exchangeable law $\mu \in \text{Pr } K^T$ is **representable** if there is a list of ingredients as above, with only the kernels κ_t allowed to depend on μ or K , such that $\mu = \kappa_{\#}^{(T)}(\mu_L \otimes \mu_L^{\otimes T_1} \otimes \mu_L^{\otimes T_2} \otimes \dots)$.*

*If an exchangeability context (T, Γ) is such that all exchangeable laws on K^T are representable for any compact metric K then we shall say that (T, Γ) **always admits representation**.*

We must stress that our chosen definition of representability is not completely canonical: although we are guided by the classical representation theorems for hypergraph-exchangeable laws and their relatives, these leading examples are sufficiently closely related one to another that it is not quite clear which features of their representation theorems we should try to keep, and which to discard, when abstracting to a more general definition. The choice we have made seems to be simple and natural, and also to reflect many of the uses to which these representation theorems are put (see [17]), but certainly it has also been selected partly because it works for what follows. An alternative formulation of the representation theorem for exchangeable arrays can be given instead in terms, for example, of sequences of auxiliary compact metric spaces Z_0, Z_1, Z_2, \dots and index sets T_1, T_2, \dots with Γ -actions $\alpha_1, \alpha_2, \dots$ from which all exchangeable laws are then obtained as pushforwards of probability measures on the product space $Z_0 \times Z_1^{T_1} \times \dots$ that are invariant under the associated overall coordinate-permuting action of Γ and have the additional property that the coordinates in Z_{i+1} are conditionally independent given the coordinates in every Z_j for $j \leq i$. The representation theorem for exchangeable arrays is treated in these terms, for example, in [6], where this choice is dictated by the use to which that theorem is then put in Section 3 of [7]; however, the formalism of representability extracted this way seems much less amenable to our needs, as well as further from the classical descriptions of Aldous and Kallenberg, and so we have settled for the above instead.

In our present terms the main Representation Theorem of Aldous, Hoover and Kallenberg for hypergraph-exchangeable laws with $T := \binom{S}{k}$ and $\Gamma := \text{Sym}_0(S)$ with its canonical action may be written as follows.

Theorem 2.3 (Representation Theorem for hypergraph-exchangeable laws). *A hypergraph-*

exchangeable law μ is representable using the data $T_i := \binom{S}{i}$ for $i \leq k$ and $T_{k+1} = T_{k+2} = \dots = \text{triv.}$, the dependency maps $\phi_i : t \mapsto \binom{t}{i}$ for $t \in \binom{S}{k}$ and $i \leq k$ and $\phi_i \equiv \emptyset$ if $i \geq k+1$, and some deterministic maps κ_t that depend on the particular choice of μ . \square

Although we have allowed arbitrary probability kernels κ_t in our present formalism, in the above concrete representation theorem (and its relatives in such works as [2, 15]) they are all deterministic maps. However, a simple transfer argument shows that this difference is purely cosmetic.

Lemma 2.4. *A (T, Γ) -exchangeable law is representable if and only if it is representable using deterministic maps $\kappa_t : [0, 1] \times [0, 1]^{\phi_1(t)} \times [0, 1]^{\phi_2(t)} \times \dots \rightarrow K$.*

Proof Clearly representability using deterministic maps amounts to a special case of representability, so we need only prove that any representable law is representable using deterministic maps. However, if we have a list of ingredients that represents μ with kernels κ_t , then by the standard Transfer Theorem (Theorem 6.10 in Kallenberg [16]) we may find deterministic maps

$$\theta_t : [0, 1] \times ([0, 1] \times [0, 1]^{\phi_1(t)} \times [0, 1]^{\phi_2(t)} \times \dots) \rightarrow K$$

such that

$$\kappa_t(x_0, \mathbf{x}_1, \dots, \cdot) = \mu_L \{ y \in [0, 1] : \theta_t(y, x_0, \mathbf{x}_1, \dots) \in \cdot \}.$$

Now, as is standard, the Lebesgue spaces $([0, 1], \mu_L)$ and $([0, 1] \times [0, 1], \mu_L \otimes \mu_L)$ are isomorphic, say via the Borel map $\xi : [0, 1] \rightarrow [0, 1]^2$, and so now defining

$$\tilde{\kappa}_t(x_0, \mathbf{x}_1, \dots) := \theta_t(\xi(x_0), \mathbf{x}_1, \dots)$$

we can check at once from the above relations that these deterministic maps also represent the original law μ . \square

3 Bauer simplices from exchangeability

We will now prove that if Γ is amenable and the exchangeability context (T, Γ) always admits representation then its simplices $\text{Pr}^\Gamma K^T$ of exchangeable laws must be Bauer for any K . We will also give a direct deduction of this Bauer property in the representative example of hypergraph exchangeability without using representability, both for completeness and because it seems interesting to compare this direct proof with arguments to prove the Poulsen property in the case of cube-exchangeability in the next section.

3.1 The Bauer property from representability

Lemma 3.1. *If Γ is amenable, and if an (T, Γ) -exchangeable probability measure $\mu \in \text{Pr}^\Gamma K^T$ is representable at all, then it is ergodic if and only if it is representable by kernels κ_t not depending on the first coordinate.*

Proof First suppose that μ is ergodic, and write it as $\kappa_\#^{(T)} \mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots}$ for some suitable family κ . Now define the family κ^u of kernels $\kappa_t^u : [0, 1] \times [0, 1]^{\phi_1(t)} \times [0, 1]^{\phi_2(t)} \times \dots \rightsquigarrow K$ by $\kappa_t^u(x_0, \mathbf{x}_1, \dots) := \kappa_t(u, \mathbf{x}_1, \dots)$ (this makes sense and is unambiguous up to equality for almost every u); clearly none of these depends on the first coordinate in $[0, 1] \times [0, 1]^{\phi_1(t)} \times \dots$, and also each $(\kappa^u)_\#^{(T)} \mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots}$ is another Γ -invariant probability on K^T such that

$$\mu = \int_0^1 (\kappa^u)_\#^{(T)} \mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots} du.$$

By the ergodicity of μ this decomposition must be trivial, and so $(\kappa^u)_\#^{(T)} \mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots} = \mu$ for almost-every u ; hence almost any of the kernel families κ^u will suffice.

Now suppose, on the other hand, that each κ_t does not depend on the first coordinate in $[0, 1] \times [0, 1]^{T_1} \times \dots$, and that $A, B \subseteq K^T$ are two Borel finite-dimensional cylinder sets, say determined by the finite sets of coordinates I, J respectively. Then by our assumption that all orbits of Γ on T and on T_i are infinite and that Γ is amenable, it follows that for some density-1 subset of $F \subset \Gamma$ we have $\phi_i(g(I)) \cap \phi_i(J) = \emptyset$ for all $g \in F$. However, this implies that κ_t and κ_s have no arguments in common for $t \in g(I)$ and $s \in J$, and so the sets $\tau^g(A)$ and B must be independent under μ . In fact this proves not only ergodicity, but even weak mixing, and we are done. \square

Proposition 3.2 (Representability implies Bauer). *If Γ is amenable and the exchangeability context (T, Γ) always admits representation then $\text{Pr}^\Gamma K^T$ is a Bauer simplex for any compact metric K .*

Proof We know that $\text{Pr}^\Gamma K^T$ is a compact convex set and that its extreme points are precisely those members that can be represented by some collection of kernels κ_t not depending on the first coordinate in $[0, 1] \times [0, 1]^{\phi_1(t)} \times \dots$; thus we need only show that if $\mu_n = (\kappa_n)_\#^{(T)} \mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots}$ are a vaguely convergent sequence of such measures then their limit μ admits a similar representation.

However, for each t the kernel κ_t defines a joining of the probability measures $\mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots}$ and $(\pi_t)_\# \mu_n$ on the product space $[0, 1] \times [0, 1]^{\phi_1(t)} \times \dots \times K$ under which the very first coordinate is independent from all the others (because κ_t does not depend on this

coordinate), and so, passing to a subsequence if necessary, we may assume that these joinings also converge to some fixed probability measure $\lambda_{\infty,t}$ on this product space. It is clear that this new measure will still have projection onto $[0, 1] \times [0, 1]^{\phi_1(t)} \times \dots$ equal to $\mu_L^{*\otimes T_1 \otimes T_2 \otimes \dots}$ and will still enjoy the independence of the first coordinate from everything else, and so if we now disintegrate these $\lambda_{\infty,t}$ over that first projection we recover kernels $\kappa_{\infty,t}$ that also do not depend on the very first coordinate and represent μ , as required. \square

Remark I do not know whether the assumption of amenability could be removed from the preceding arguments. \triangleleft

3.2 The Bauer property in the particular context of hypergraph exchangeability

Before moving on, let us include a second proof that the classical hypergraph-exchangeability context has the Bauer property that uses only a very elementary property enjoyed by that context, rather than the representation theorem. This subsection is not essential to the main thread of this note, but is included mainly to advertise the question of whether the argument that it contains can be generalized further.

Definition 3.3 (Distant multiple transitivity). *We shall write that an exchangeability context (T, Γ) is **distantly multiply transitive (DMT)** if for any finite $I, J \subset T$ there is some subset $E \subseteq \Gamma$ of density 1 and such that for any $\gamma_1, \gamma_2 \in E$ there is some $\xi \in \Gamma$ with $\xi \upharpoonright_I = \text{id}_I$ and $\xi \circ \gamma_1 \upharpoonright_J = \gamma_2 \upharpoonright_J$.*

It is immediate to check that the hypergraph exchangeability context is DMT, and so the following result applies to that context in particular.

Proposition 3.4 (DMT implies Bauer). *If Γ is amenable and (T, Γ) is DMT then it has the Bauer property.*

Proof We follow closely the analogous argument of Glasner and Weiss in [9]. Suppose that Γ is amenable, that (T, Γ) is DMT, that $\mu \in \text{Pr}^\Gamma K^T$ can be vaguely approximated by ergodic measures, and that $A \in \Sigma_{K^T}$ is invariant with $a := \mu(A) \in [0, 1]$. For any $\varepsilon > 0$ there are a finite set $J \subset T$ and a continuous function $f : K^J \rightarrow [0, 1]$ such that $\|1_A - f \circ \pi_J\|_{L^1(\mu)} < \varepsilon$, and hence $\int_{K^T} f \circ \pi_J d\mu \approx_\varepsilon a$. From the invariance of A it follows that we actually have $\|1_A - f \circ \pi_J \circ \tau^\gamma\|_{L^1(\mu)} < \varepsilon$ for any $\gamma \in \Gamma$.

Now, since (T, Γ) is DMT and J is finite, there is some $E \subseteq \Gamma$ with asymptotic density 1 such that for any $\gamma_1, \gamma_2 \in E$ there is some $\xi \in \Gamma$ such that $\xi \upharpoonright_J = \text{id}_J$, and so $f \circ \pi_J \circ \tau^\xi =$

$f \circ \pi_J$, whereas $\xi \circ \gamma_1 \upharpoonright_J = \gamma_2 \upharpoonright_J$ and so $f \circ \pi_J \circ \tau^{\gamma_1} \circ \tau^\xi = f \circ \pi_J \circ \tau^{\gamma_2}$. Let us now fix some representative member $\gamma_0 \in E$.

Next, since $f \circ \pi_J$ and $(f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^{\gamma_0})$ are continuous, by assumption we can always find some ergodic $\mu' \in \text{Pr}^\Gamma K^T$ with

$$\int_{K^T} f \circ \pi_J \, d\mu' \approx_\varepsilon \int_{K^T} f \circ \pi_J \, d\mu$$

and

$$\int_{K^T} (f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^{\gamma_0}) \, d\mu' \approx_\varepsilon \int_{K^T} (f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^{\gamma_0}) \, d\mu.$$

Letting $(I_n)_{n \geq 1}$ be a Følner sequence in Γ , it follows from the ergodicity of μ' that

$$\frac{1}{|I_N|} \sum_{\gamma \in I_N} \int_{K^T} (f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^\gamma) \, d\mu' \rightarrow \left(\int_{K^T} f \circ \pi_J \, d\mu' \right)^2 \approx_{2\varepsilon} \left(\int_{K^T} f \circ \pi_J \, d\mu \right)^2 \approx_{2\varepsilon} a^2$$

as $N \rightarrow \infty$. On the other hand, we know that for N sufficiently large at least $(1 - \varepsilon)$ -proportion of $\gamma \in I_N$ lie in $E \cap I_N$, and that $\gamma_0 \in E \cap I_N$, and so by choosing a suitable ξ they must all give exactly the same value for $\int_{K^T} (f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^\gamma) \, d\mu'$; and therefore for N sufficiently large we must also have

$$\begin{aligned} \frac{1}{|I_N|} \sum_{\gamma \in I_N} \int_{K^T} (f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^\gamma) \, d\mu' &\approx_\varepsilon \int_{K^T} (f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^{\gamma_0}) \, d\mu' \\ &\approx_\varepsilon \int_{K^T} (f \circ \pi_J) \cdot (f \circ \pi_J \circ \tau^{\gamma_0}) \, d\mu \approx_{2\varepsilon} \int_{K^T} 1_A \cdot 1_A \, d\mu = \mu A = a. \end{aligned}$$

Combining these approximations shows that $a \approx_{6\varepsilon} a^2$ for any $\varepsilon > 0$, and so in fact we must have $a \in \{0, 1\}$, and μ must itself be ergodic. \square

4 The Poulsen property for cube-exchangeable measures

We will now show that, quite unlike the cases studied in the previous two sections, if (T, Γ) is the cube-exchangeability context (and K is nontrivial) then $\text{Pr}^\Gamma K^T$ is actually the Poulsen simplex. This argument is also closely motivated by that of Glasner and Weiss in [9], where they show that in the case of the exchangeability context (Γ, R_Γ) comprising a group Γ and its right-regular representation on itself, the simplex $\text{Pr}^\Gamma \{0, 1\}^\Gamma$ of invariant probability measures is either Bauer or Poulsen precisely according as Γ has

or fails Kazhdan's property (T). No condition like property (T) will enter our analysis — indeed, the groups of immediate interest to us are all locally finite, hence trivially amenable — but we will follow closely the basic steps of their construction.

There are essentially two of these steps. We first show that in case $K = \{0, 1\}$ the particular example $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ of a non-ergodic member of $\text{Pr}^\Gamma\{0, 1\}^T$ is vaguely approximable by members that are not only ergodic, but actually weakly mixing; and then we use this fact through the construction of a certain joining to show that quite generally whenever μ_1 and μ_2 in $\text{Pr}^\Gamma K^T$ are approximable by ergodic measures, so is their average $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. We need to ensure weak mixing in the first step because we shall need to ensure the ergodicity of a certain product in the second, but this makes little difference to the other details of the proofs. It is easy to see that this then implies the Poulsen property.

Lemma 4.1. *Let $(T, \Gamma) = (\mathbb{F}_2^{\oplus \mathbb{N}}, \text{Isom } \mathbb{F}_2^{\oplus \mathbb{N}})$. Then the measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \in \text{Pr}^\Gamma\{0, 1\}^T$ is vaguely approximable by weakly mixing members of $\text{Pr}^\Gamma\{0, 1\}^T$.*

Proof We need to show that for any $\varepsilon > 0$ and $N \geq 1$ there is some strongly mixing measure $\mu \in \text{Pr}^\Gamma\{0, 1\}^T$ such that both

$$\mu\{\omega \in \{0, 1\}^T : \omega \upharpoonright_{\mathbb{F}_2^N} = \mathbf{0}\} \geq \frac{1}{2} - \varepsilon$$

and

$$\mu\{\omega \in \{0, 1\}^T : \omega \upharpoonright_{\mathbb{F}_2^N} = \mathbf{1}\} \geq \frac{1}{2} - \varepsilon.$$

There are many possible ways to construct such a μ ; the following seems to be one of the simplest. We specify μ as the law of the member of $\{0, 1\}^T$ output by the following random procedure. For any $p \in [0, 1]$ let ν_p be the product measure on $\mathbb{F}_2^{\mathbb{N}}$ with $\nu_p\{z : z_i = 1\} = p$ for every $i \in \mathbb{N}$; and for any $z = (z_i)_{i \in \mathbb{N}} \in \mathbb{F}_2^{\mathbb{N}}$ and $x \in \mathbb{F}_2^{\oplus \mathbb{N}}$ define $\langle x, z \rangle := \sum_{i \in \mathbb{N}} x_i z_i \pmod{2}$ (this sum being actually always finite). Now let μ be the law of the characteristic function of the random subset $\{x \in \mathbb{F}_2^{\oplus \mathbb{N}} : \langle x, z \rangle + \eta = 0 \pmod{2}\}$ where $z \sim \nu_p$ for some very small $p > 0$ and $\eta \in \mathbb{F}_2$ is chosen independently and uniformly at random.

It is clear that this μ is Γ -invariant and strongly mixing provided $p \neq 0$, but if p is very small then for our chosen N we have $\nu_p\{z : z_1 = z_2 = \dots = z_N = 0\} \geq 1 - \varepsilon$, and conditioned on the event $\{z : z_1 = z_2 = \dots = z_N = 0\}$ we must have also

$$1_{\{x \in \mathbb{F}_2^{\oplus \mathbb{N}} : \langle x, z \rangle + \eta = 0 \pmod{2}\}} = \begin{cases} 1 & \text{if } \eta = 0 \quad (\text{occurs with prob. } \frac{1}{2}) \\ 0 & \text{if } \eta = 1 \quad (\text{occurs with prob. } \frac{1}{2}), \end{cases}$$

which proves the desired vague approximation to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. □

Theorem 4.2. *The cube-exchangeability context $(T, \Gamma) = (\mathbb{F}_2^{\oplus \mathbb{N}}, \text{Isom } \mathbb{F}_2^{\oplus \mathbb{N}})$ has the Poulsen property.*

Proof Let K be any compact metric space containing at least two points. As argued by Glasner and Weiss in [9], it suffices to prove that for any two ergodic $\mu_1, \mu_2 \in \text{Pr}^\Gamma K^T$, their average $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ can be approximated by ergodic members of $\text{Pr}^\Gamma K^T$; for then it follows by repeated approximation that the ergodic probability measures must be dense in their own convex hull, but this is the whole of $\text{Pr}^\Gamma K^T$.

Thus, it is enough to show that for any $\varepsilon > 0$ and finite list of continuous functions $f_1, f_2, \dots, f_m : K^T \rightarrow [0, 1]$ there is some ergodic $\mu \in \text{Pr}^\Gamma K^T$ such that

$$\int_{K^T} f_i d\mu \approx_{2\varepsilon} \frac{1}{2} \int_{K^T} f_i d\mu_1 + \frac{1}{2} \int_{K^T} f_i d\mu_2 \quad \forall i \leq m.$$

Moreover, by the Stone-Weierstrass Theorem we may assume each f_i depends only on coordinates in some fixed finite subset $J \subset T$, and so may factorize and rewrite it as $f_i \circ \pi_J$.

First, let us choose $\mu_0 \in \text{Pr}^\Gamma \{0, 1\}^T$ weakly mixing and satisfying $\mu_0(A) \approx_\varepsilon \frac{1}{2}\delta_0(A) + \frac{1}{2}\delta_1(A)$ for all $A \subseteq \{0, 1\}^T$ depending only on coordinates in J ; this is possible by Lemma 4.1. Now consider any ergodic cube-exchangeable joining λ of the two measures μ_1 and μ_2 on the product space $(K^2)^T$ (such can be obtained, for example, by taking any ergodic component of the simple product $\mu_1 \otimes \mu_2$), and now from this construct the product $\mu_0 \otimes \lambda$, a member of $\text{Pr}^\Gamma(\{0, 1\} \times K^2)^T$. Since μ_0 is weakly mixing, this product is still ergodic.

We now complete the proof by specifying a Γ -equivariant map $\psi : (\{0, 1\} \times K^2)^T \rightarrow K^T$ whose law as a K^T -valued random variable under $\mu_0 \otimes \lambda$ will be the ergodic approximating measure that we seek: given a point $(\eta, \omega^{(1)}, \omega^{(2)}) \in (\{0, 1\} \times K^2)^T$, we define $\psi(\eta, \omega^{(1)}, \omega^{(2)})_t$ to be $\omega_t^{(1)}$ if $\eta_t = 0$, and $\omega_t^{(2)}$ if $\eta_t = 1$. Let us also write $\psi^{(1)}$ and $\psi^{(2)}$ for the usual projection maps $(\{0, 1\} \times K^2)^T \rightarrow K^T$ onto the first and second copies of K^T respectively.

It is clear that this ψ is equivariant, and that its law $\psi_\#(\mu_0 \otimes \lambda)$ must, like $\mu_0 \otimes \lambda$, be

ergodic. Finally,

$$\begin{aligned}
\int_{K^T} f_i \circ \pi_J \, d\psi_{\#}(\mu_0 \otimes \lambda) &= \int_{(\{0,1\} \times K^2)^T} f_i \circ \pi_J \circ \psi \, d(\mu_0 \otimes \lambda) \\
&= \int_{\{\eta \upharpoonright_J = \mathbf{0}\}} f_i \circ \pi_J \circ \psi \, d(\mu_0 \otimes \lambda) + \int_{\{\eta \upharpoonright_J = \mathbf{1}\}} f_i \circ \pi_J \circ \psi \, d(\mu_0 \otimes \lambda) \\
&\quad + \int_{\{\eta \upharpoonright_J = \mathbf{0}\}^c \cap \{\eta \upharpoonright_J = \mathbf{1}\}^c} f_i \circ \pi_J \circ \psi \, d(\mu_0 \otimes \lambda) \\
&\approx_{\varepsilon} \mu_0\{\eta \upharpoonright_J = \mathbf{0}\} \cdot \int_{(K^2)^T} f_i \circ \pi_J \circ \psi^{(1)} \, d\lambda + \mu_0\{\eta \upharpoonright_J = \mathbf{1}\} \cdot \int_{(K^2)^T} f_i \circ \pi_J \circ \psi^{(2)} \, d\lambda \\
&\approx_{\varepsilon} \frac{1}{2} \int_{K^T} f_i \circ \pi_J \, d\mu_1 + \frac{1}{2} \int_{K^T} f_i \circ \pi_J \, d\mu_2,
\end{aligned}$$

where we have deduced from the known quality of our approximation $\mu_0 \approx \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ that

$$\mu_0\{\eta \upharpoonright_J = \mathbf{0}\}, \mu_0\{\eta \upharpoonright_J = \mathbf{1}\} \approx_{\varepsilon} \frac{1}{2}$$

and

$$\mu_0(\{\eta \upharpoonright_J = \mathbf{0}\}^c \cap \{\eta \upharpoonright_J = \mathbf{1}\}^c) \approx_{\varepsilon} 0.$$

This completes the proof. \square

Corollary 4.3 (Failure of cube-exchangeable representability). *For the infinite discrete cube context (T, Γ) , the exchangeable laws $\text{Pr}^{\Gamma}[0, 1]^T$ do not admit representation.*

Proof This follows at once from Theorem 4.2 and Proposition 3.2. \square

5 Some further questions

5.1 Further analysis of cube-exchangeable measures

In [4] (Examples 16.7 and 16.10) Aldous introduces an interesting family of examples of cube-exchangeable measures built from reversible random walks on a compact Abelian group, and asks whether these might play a rôle in a more complete representation theorem for such measures. Since they do not seem to fall easily into the framework set up in Section 2, it would be remiss of us not to mention them separately.

Letting U be such a group endowed with its Borel σ -algebra Σ_U and Haar measure μ_U , and suppose also that $\nu \in \text{Pr } U$. From this data we can define a measure μ on $U^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ as the law of the following randomized selection of a point $(g_v)_{v \in \mathbb{F}_2^{\oplus \mathbb{N}}}$ of this space:

- First select $g_0 \in U$ uniformly at random;
- Now select $g_i^\circ \in U$ for each $i \in \mathbb{N}$ independently at random with law ν , and let $g_v := g_0 + \sum_{i \in \mathbb{N}} v_i g_i^\circ$ for all $v = (v_i)_{i \in \mathbb{N}} \in \mathbb{F}_2^{\oplus \mathbb{N}}$.

The $\text{Sym}_0(\mathbb{N})$ -symmetry (‘hypergraph-exchangeability’) of this law μ is manifest; in order to guarantee full cube-exchangeability it turns out to be necessary and sufficient that ν satisfy the symmetry condition that the two maps $(g_0, g_1) \mapsto (g_0, g_0 + g_1)$ and $(g_0, g_1) \mapsto (g_0 + g_1, g_0)$ have the same law under the product measure $\mu_U \otimes \nu_0$.

Notice that we have already met one of these Abelian group examples in the form of the measure μ constructed from ν_p during the proof of Lemma 4.1.

Cube-exchangeable systems of this form (or, more generally, factors of such systems) are surely rather special, but they fit into a considerably more general framework, and this may afford some greater purchase over the general case. Let us approach this generalization from a rather different direction.

Since U is an Abelian group we may describe a general point of $U^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ using a Möbius inversion formula: for any $(g_v)_{v \in \mathbb{F}_2^{\oplus \mathbb{N}}} \in U^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ there are unique $(u_\alpha)_{\alpha \in \binom{\mathbb{N}}{<\infty}} \in U^{\binom{\mathbb{N}}{<\infty}}$ such that

$$g_v = \sum_{\alpha \subseteq v^{-1}\{1\}} u_\alpha = \sum_{\alpha \in \binom{\mathbb{N}}{<\infty}} \left(\prod_{i \in \alpha} v_i \right) u_\alpha \quad \forall v \in \mathbb{F}_2^{\oplus \mathbb{N}},$$

and it is routine to check that the resulting bijection $\Phi : U^{\mathbb{F}_2^{\oplus \mathbb{N}}} \rightarrow U^{\binom{\mathbb{N}}{<\infty}}$ is actually a homeomorphism, and that it is covariant for the coordinate-permuting actions of $\text{Sym}_0(\mathbb{N})$ on the domain and on the target. It follows that any hypergraph-exchangeable $\mu \in \text{Pr}^{\text{Sym}_0(\mathbb{N})} U^T$ is pushed forward by Φ to a hypergraph-exchangeable measure $\Phi_\# \mu$ on $U^{\binom{\mathbb{N}}{<\infty}}$, and indeed that this gives an affine homeomorphism between the simplices of hypergraph-exchangeable measures. However, the stronger assumption that μ be cube-exchangeable is then converted under Φ into a rather larger set of additional symmetries for $\Phi_\# \mu$, and these are not obviously easier to describe explicitly than the original cube-exchangeable structure of μ .

Indeed, if $\mu \in \text{Pr}^\Gamma K^T$ for an arbitrary compact metric space K and the one-dimensional marginals $(\pi_v)_\# \mu \in \text{Pr} K$ (which must all agree) are atomless, then we can simply choose any non-discrete compact Abelian group U and a function $(K, (\pi_0)_\# \mu) \rightarrow (U, \mu_U)$ that defines a measure-algebra-isomorphism and observe that applying this function pointwise gives an isomorphism from $\Gamma \curvearrowright K^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ to $\Gamma \curvearrowright U^{\mathbb{F}_2^{\oplus \mathbb{N}}}$, and so without any additional assumptions the above examples of cube-exchangeable laws on Abelian groups lose no generality at all. However, we might ask whether we can find a route to a more interesting representation theorem through a canny choice of the isomorphism $(K, (\pi_0)_\# \mu) \rightarrow (U, \mu_U)$,

for which the additional constraints on the joint law of $(u_\alpha)_{\alpha \in \binom{[N]}{<\infty}}$ can then be described explicitly. A little more generally, can we some U and some cube-exchangeable measure θ on $U^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ of an especially simple form such that μ is a coordinatewise factor of θ , say $\mu = (f^{\mathbb{F}_2^{\oplus \mathbb{N}}})_{\#} \theta$ for some Borel $f : U \rightarrow K$. For example, can we choose a θ under which the summands in the Möbius inversion formula corresponding to sets of different sizes are independent?

We will not offer so much here, but merely note that more can be said in certain simple cases. For example, if $u_\alpha = 0$ a.s. whenever $|\alpha| \geq 2$, then the above laws μ must be measures of the kind described in Aldous' example, as may be checked by hand from the rank-2 case of the hypergraph-exchangeability representation theorem applied to $\Phi_{\#} \mu$.

More generally, we can focus attention on the sub-simplices of cube-exchangeable laws that are concentrated on certain Γ -invariant closed subsets of K^T . For each $r \geq 1$ let Ω_r be the subset of those $g \in U^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ with the property that ‘all r -faces sum to zero’:

$$g \in \Omega_r \quad \Leftrightarrow \quad \sum_{v \in F} g_v = 0 \quad \text{for each } r\text{-face } F \subseteq \mathbb{F}_2^{\oplus \mathbb{N}}.$$

This suggestion is made by Aldous in [4] (example 16.20), where he also points out that some such restricted measures already defeat any overly-simple approach to a representation theorem for cube-exchangeability using group random walks.

It is easy to check that concentration on Ω_2 is equivalent to the abovementioned condition that $u_\alpha = 0$ a.s. whenever $|\alpha| \geq 2$. It turns out that in the special case $U = \mathbb{F}_2$ the points of Ω_r have a particularly simple explicit description: in this case, identifying $U^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ as the space of functions $\mathbb{F}_2^{\oplus \mathbb{N}} \rightarrow \mathbb{F}_2$, an explicit calculation of the Möbius inversion gives at once that a function $g : \mathbb{F}_2^{\oplus \mathbb{N}} \rightarrow \mathbb{F}_2$ lies in Ω_r if and only if it is a polynomial of degree at most r . (Note that for a general field \mathbb{K} it is fairly straightforward to prove that those functions $f : \mathbb{K}^d \rightarrow \mathbb{K}$ that have zero sum across any *affine* copy of the r -dimensional discrete cube in \mathbb{K}^d must be a polynomial of degree at most r , for any underlying field \mathbb{K} . However, under the present weaker assumption of zero-sums across only *isometric* copies of the r -cube in \mathbb{F}_2^d , and it is not hard to find examples showing that the implication of degree- r polynomiality follows only over the smallest field \mathbb{F}_2 .)

5.2 The geometry of subsimplices and relations to property testing

Theorem 4.2 has consequences for the relations between the vague topology and the ‘ \bar{d} ’- (or joining) topology (considered by Aldous in the case of hypergraph exchangeability

in [3] and Section 15 of [4]). This latter is defined by the \bar{d} -**metric** ρ on exchangeable probability measures, given by

$$\rho(\mu, \nu) := \inf_{\lambda \in J(\mu, \nu)} \lambda\{(\omega, \eta) \in K^T \times K^T : \omega_v \neq \eta_v\}$$

for any (arbitrary) choice of reference index $v \in T$, where $J(\mu, \nu)$ denotes the collection of all **joinings** of μ and ν : Γ -invariant probability measures on $K^T \times K^T$ having first marginal μ and second marginal ν . If $\rho(\mu, \nu)$ is small we shall write informally that μ and ν have a **near-diagonal joining**.

The joining topology is clearly at least as strong as the vague topology, and in general it is strictly stronger (see [3], for example). However, given a Γ -invariant closed subset $\Omega \subseteq K^T$, we can consider the subsimplex $\text{Pr}^\Gamma \Omega \subseteq \text{Pr}^\Gamma K^T$ of exchangeable measures concentrated on Ω , and ask whether the two different neighbourhood bases of this subsimplex defined by these two topologies might coincide. This question is motivated by the case of hypergraph-exchangeability, for which it can be proved that these bases *do* always coincide; this follows, in particular, from the rather more precise results for such closed subsets contained in [7]. However, by making reference to the Poulsen property, we can see that this is not always the case for cube-exchangeability.

Proposition 5.1. *If an exchangeability context (T, Γ) has the Poulsen property and these two neighbourhood bases around $\text{Pr}^\Gamma \Omega$ are equivalent then $\text{Pr}^\Gamma \Omega$ must also be the Poulsen simplex.*

Proof In general, if μ_1 is ergodic and is close to μ_2 in the vague topology, it need not follow that μ_1 is close to any of the ergodic components of μ_2 in the vague topology. However, if in fact μ_1 is joining-close to μ_2 then it does follow that it is joining-close to many of the ergodic components of μ_2 , by considering the ergodic decomposition of the joining itself.

Let the situation be as described, and suppose that $\mu \in \text{Pr}^\Gamma \Omega$; we must show that μ is vaguely approximable by extreme points of $\text{Pr}^\Gamma \Omega$. Since $\text{Pr}^\Gamma \Omega$ is just the subset of those members of $\text{Pr}^\Gamma K^T$ that are concentrated on Ω , its extreme points are still just its ergodic members.

By the Poulsen property of $\text{Pr}^\Gamma K^T$, we know μ can be vaguely approximated by ergodic measures in this larger simplex. On the other hand, by the assumed equivalence of the two neighbourhood bases, it follows that provided these approximating measures are close enough to the subsimplex $\text{Pr}^\Gamma \Omega$ for the vague topology, they actually have near-diagonal joinings with members of this smaller simplex $\text{Pr}^\Gamma \Omega$.

However, if $\mu_1 \in \text{Pr}^\Gamma K^T$ is ergodic and $\lambda \in \text{Pr}^\Gamma(K^T \times K^T)$ is a near-diagonal joining of μ_1 to some member of $\text{Pr}^\Gamma \Omega$, then the components of the ergodic decomposition of λ must

(almost surely) be joinings of μ_1 to ergodic measures that are still members of $\text{Pr}^\Gamma \Omega$, and in order that λ be near-diagonal these ergodic components of λ must also be near-diagonal with high probability. It follows that μ_1 must actually be joining-close, and hence vaguely close, to some ergodic members of $\text{Pr}^\Gamma \Omega$; and since μ_1 was itself vaguely close to μ , we deduce that μ must be vaguely approximable by extreme points of $\text{Pr}^\Gamma \Omega$, as required. \square

We suspect that the above implication cannot be reversed (in that there are also Ω for which the neighbourhood bases do not coincide, but for which $\text{Pr}^\Gamma \Omega$ is Poulsen anyway).

Corollary 5.2. *The subset $\Omega_2 \subseteq \mathbb{F}_2^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ is such that the joining neighbourhood basis of the simplex $\text{Pr}^\Gamma \Omega_2$ is strictly stronger than the vague neighbourhood basis.*

Proof By the previous proposition, it suffices to argue that $\text{Pr}^\Gamma \Omega_2$ is not Poulsen; however, as discussed in the previous subsection, the members of $\text{Pr}^\Gamma \Omega_2$ are precisely Aldous' random walk examples in the case $U = \mathbb{F}_2$, and it is now easy to check from this that the simplex in question has set of extreme points precisely the measures μ constructed from ν_p for different $p > 0$ from the proof of Lemma 4.1, together with δ_0 and δ_1 , and that this set of extreme points has only the one additional non-ergodic cluster point $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ (indeed, that lemma itself guarantees that this must be cluster point; it is the argument of Theorem 4.2 that then necessarily takes us outside $\text{Pr}^\Gamma \Omega_2$, and so does not apply to this sub-simplex). Thus, $\text{Pr}^\Gamma \Omega_2$ cannot be Poulsen. \square

In the setting of hypergraph exchangeability, it turns out that there is a close relationship between properties of the sub-simplex $\text{Pr}^T \Omega$ and of the conditions on a point of K^T needed to guarantee membership of Ω . In addition, it turns out that this latter membership condition can be identified simply with some hereditary property of K -colourings of finite hypergraphs (precisely, so that a point of K^T lies in Ω if and only if when regarded as a K -coloured hypergraph all of its finite induced coloured sub-hypergraphs have that hereditary property). From this vantage point, a suitable analysis of this simplex can be converted into a proof that all such properties are 'efficiently testable' (following essentially a translation of older, purely combinatorial arguments to that effect; see, in particular, Alon and Shapira [5] and Rödl and Schacht [18]). We shall not enter into these notions further here, but refer the reader to the complete account in [7].

It seems clear that a similar notion of efficient testability can be formulated in the setting of discrete cubes and their isometries: in general, we would write that a property \mathcal{P} of all subsets of faces of the finite discrete cubes \mathbb{F}_2^N is **testable** if for any $\varepsilon > 0$ there are some $N(\varepsilon) \geq J(\varepsilon) \geq 1$ and $\delta(\varepsilon) > 0$ such that, if $N \geq N(\varepsilon)$ and $E \subseteq \mathbb{F}_2^N$, and if we know that a $J(\varepsilon)$ -face F of \mathbb{F}_2^N chosen uniformly at random has probability at least $1 - \delta(\varepsilon)$ of having $F \cap E \in \mathcal{P}$, then there is some $E' \subseteq \mathbb{F}_2^N$ having $E' \in \mathcal{P}$ and $|E \Delta E'| < \varepsilon 2^N$.

Although we are not aware of a rigorous relationship between the question of Proposi-

tion 5.1 and testability, by analogy with the results of [7] we suspect from that Proposition that the property Ω_2 is *not* testable; and in fact a direct re-write of the particular infinitary proofs we have given in finitary terms in a high-dimensional cube \mathbb{F}_2^N shows that this is so; we omit the details.

5.3 Affine transformations of the infinite-dimensional discrete cube

We have already discussed cube-exchangeability as a strengthening of the condition of hypergraph-exchangeability treated by classical exchangeability theory. However, it may be worth recalling that an even stronger exchangeability context on $T = \mathbb{F}_2^{\oplus \mathbb{N}}$ has also appeared implicitly in a number of recent works, with Γ the group of all affine transformations of T .

In particular, this setting closely relates to several questions of current interest in arithmetic combinatorics concerning the counting of affine copies of various patterns (such as finite-dimensional cubes) in subsets of \mathbb{F}_2^N for large N . These questions often correspond naturally to descriptions of probability measures on $\{0, 1\}^{\mathbb{F}_2^{\oplus \mathbb{N}}}$ that are $\text{Aff } \mathbb{F}_2^{\oplus \mathbb{N}}$ -invariant via a suitable correspondence principle, analogous to the well-known Furstenberg correspondence principle relating subsets of \mathbb{Z} to measure-preserving \mathbb{Z} -actions (see, for example, Furstenberg's book [8]). Closely-related to this line of research is the investigation of the 'Gowers-inverse conjecture' of Green and Tao in the case of the vector spaces \mathbb{F}_2^N , which are phrased in terms of correlations of individual \mathbb{C} -valued functions on \mathbb{F}_2^N with functions of certain special forms. However, this conjecture has recently been shown to fail in general in this setting in the paper [10] of Green and Tao, and so some more complicated kinds of ingredient seem to be required for such a structure theorem.

In our more infinitary set-up, we suspect that in the presence of this rather stronger symmetry a much more detailed analysis of the structure of the exchangeable measures is possible, and that such an analysis will probably rely on more ergodic-theoretic tools (such as those developed for the proof or convergence and expression of the limit of nonconventional ergodic averages in the case of \mathbb{Z} -systems; see, in particular, the works of Host & Kra [14] and Ziegler [?]); however, we have not investigated this possibility further. We also direct the reader to Subsection 4.7 of [6] for a very informal discussion of the different approaches to the extraction of structural information for invariant measures in the study of exchangeability, on the one hand, and ergodic theory on the other.

5.4 The Poulsen property for other exchangeability contexts

We suspect that the conclusion of Theorem 4.2 holds much more generally: that for an amenable group Γ it is only in the presence of some very special exchangeability context (such as those that are DMT) that the Poulsen property fails.

Is it possible to formulate a more general condition under which an exchangeability context has the Poulsen property that will subsume Theorem 4.2? On the other hand, is there some condition related to that of being DMT that is actually equivalent to the Bauer property (possibly only for amenable Γ)? Can the simplex $\text{Pr}^\Gamma K^T$ ever be neither Bauer nor Poulsen?

5.5 Cube-exchangeability for finer-grained cubes

We suspect that the results of this paper extend to the analogous definition of exchangeability on the finer-grained cubes $(\mathbb{Z}/m\mathbb{Z})^{\oplus \mathbb{N}}$ for $m > 2$ (indeed, the situation there is surely even more wild, if anything), but it is not clear whether these exhibit any additional new phenomena.

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